

CURVE DIFFUSION AND STRAIGHTENING FLOWS ON PARALLEL LINES

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ABSTRACT. In this paper, we study families of immersed curves $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ with free boundary supported on parallel lines $\{\eta_1, \eta_2\} : \mathbb{R} \rightarrow \mathbb{R}^2$ evolving by the curve diffusion flow and the curve straightening flow. The evolving curves are orthogonal to the boundary and satisfy a no-flux condition. We give estimates and monotonicity on the normalised oscillation of curvature, yielding global results for the flows.

1. INTRODUCTION

Fourth-order extrinsic curvature flow have recently enjoyed considerable attention in the literature. Two model flows are the surface diffusion flow, where points move with velocity $\Delta^\perp \vec{H}$, and the Willmore flow, where points move with velocity $\Delta^\perp \vec{H} + \vec{H} |A^\circ|^2$. These curvature flow are one-parameter families of surfaces immersed in \mathbb{R}^3 via immersions $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^3$, with \vec{H} the mean curvature vector, Δ^\perp the Laplacian on the normal bundle along f , and A° the tracefree second fundamental form.

Surface diffusion flow, proposed by Mullins [46] in 1956, arises as a model for several phenomena [9, 59]. As such it has received and continues to receive intense attention from the applied mathematics community. Global analysis for the surface diffusion flow is restricted at the moment to special situations, and although the theory of singularities for the flow has received some attention [64, 65] it is far from well-understood. The surface diffusion flow is variational, being the H^{-1} -gradient flow for the area functional. The Willmore flow is also variational, being the steepest descent L^2 -gradient flow for the Willmore functional. The Willmore functional is, up to normalisation, the integral of the mean curvature \vec{H} squared. A prototypical bending energy, it has been argued that the Willmore functional was considered first by Sophie Germain in the early 19th century. The Willmore functional drew significant interest from Blaschke [4, 5, 6] and his school, including Thomsen and Schadow, who first presented the Euler-Lagrange operator. Their interest in the Willmore functional stems from its invariance under the Möbius group of \mathbb{R}^3 (so long as inversions are not centred on the surface, see [2, 3, 11, 32] for example for a precise formula). This invariance lies at the heart of many of its applications, both to physics and back to mathematics itself, for example in embedding problems. The appeal of the functional is so universal that the Willmore conjecture [68], asserting that the global minimiser among surfaces in \mathbb{R}^3 with genus one is achieved by the Clifford torus (and closed conformal images thereof), generated significant attention (a selection is [10, 37, 52, 55]), before being recently solved in a breakthrough work [42]. The Willmore flow was first studied by Kuwert and Schätzle [33, 34, 35] who set up a general framework

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that is by now a standard methodology used to understand large varieties of higher-order curvature flow. Applications and modifications of this framework exist for the surface diffusion flow [65, 62], the geometric triharmonic heat flow [44], and polyharmonic flows [50].

Although in some special cases maximum-principle style results hold, more typical is a kind of ‘almost’ maximum principle, and an ‘eventual’ positivity, see [16, 22, 25, 26] for the parabolic and [27] and for the elliptic settings respectively). Many of the tools and techniques used in the analysis of second-order curvature flow can not be applied to the study of fourth and higher-order curvature flow. In addition to the development of new techniques, it is a natural focus of research effort to determine the extent to which modifications of known techniques apply to various fourth-order curvature flow in different scenarios. This is where the present paper fits into the picture. We treat the one-dimensional case for the surface diffusion and Willmore flows with free boundary, called the *curve diffusion flow* and *elastic flow* (or curve lengthening/straightening flow) respectively.

In order to differentiate easily between these three flows, we label them as follows:

- (CD) Curve diffusion flow
- (E) Elastic flow

The main results, Theorem 1.2 for (CD) and Theorem 1.4 for (E), consider the question of geometric stability, where closeness to an equilibrium is measured explicitly in terms of a geometric quantity. We also present some conjectures and a question on a suitable adaptation of Proposition 1.5 from [62]. This directly addresses for (CD) the question of preservation of positivity raised above by measuring the total amount of time during which a global solution may remain not strictly graphical. The evolving families of curves we study have *free boundary*, supported on parallel lines in the plane (see Figure 2).

Second-order curvature flow with free boundary have been considered since the 90s [51, 56, 57, 58] and continues to receive significant research attention (for a sample of the growing literature, see [8, 17, 31, 36, 40, 41, 45, 60, 61, 63, 66, 67]). Fourth-order curvature flow with various boundary conditions have received some recent attention, with work particularly relevant to this paper in [13, 14, 15, 23, 24, 38, 39, 47, 49]. In [23, 24] stability results are proved for curves evolving by (CD) that are graphical and nearby equilibria (with closeness measured in terms of height and $\|k_s\|_2^2$) evolving in bounded domains with free boundary. Although our setting is fundamentally parametric and therefore distinct, our results here, for the curve diffusion flow, can be thought of as naturally complementing these. The evolving curves considered in this paper are supported on straight lines, so the analogue of ‘domain’ from [23, 24] is always unbounded. We consider immersed curves, with possibly self-intersecting image. Intersections in the image may result from the curve touching itself, or from the curve intersecting one of the straight supporting lines. This allows global results for perturbations of arcs of multiply-covered circles for instance. Considering curves supported on parallel lines allows for results on unbounded, cocompact initial data as well. As the supporting curves are parallel, repeated reflection produces an entire curve.

Stability for the elastic flow is a classically difficult problem. The flow (E) is the steepest descent L^2 -gradient flow for the elastic energy:

$$E(\gamma) = \int_{\gamma} k^2 ds,$$

where $\gamma : [-1, 1] \rightarrow \mathbb{R}^2$ is a smooth immersed plane curve, k its scalar curvature and ds the arclength element. This energy is *not* scale-invariant, and can be decreased by enlarging the curve through homothety. Circles and curves with constant curvature are not equilibria; they are expanders.

There exist infinitely many straight line segments in that are stationary under the flow. Despite this it seems difficult to imagine that the flow (E) without a constraint would be stable, especially without imposing an additional symmetry condition, as glued in arcs of circles would still prefer to expand under the flow. In fact, if the distance between the parallel lines $|e|$ is zero, then circles expand. By slowly separating the two lines (continuously increasing $|e|$ for example) and using a continuous dependence on data result in appropriate spaces, there seems to exist many non-compact trajectories for the flow. With this in mind, stability of the straight line under (E) seems unlikely. Nevertheless we do achieve stability for (E) without needing to resort to a length constraint. This argument requires an initial condition.

Let us formally introduce the evolution equations. Suppose $\gamma : [-1, 1] \rightarrow \mathbb{R}^2$, $\eta_i : \mathbb{R} \rightarrow \mathbb{R}^2$ ($i = 1, 2$) are regular smooth immersed plane curves such that γ meets η_i perpendicularly with zero flux at its endpoints; that is,

$$(1) \quad \gamma(-1) \in \eta_1(\mathbb{R}), \quad \gamma(1) \in \eta_2(\mathbb{R}), \quad \langle \nu, \nu_{\eta_i} \rangle(\pm 1) = 0, \quad k_s(\pm 1) = 0.$$

Above we have used ν to denote a unit normal vector field on γ , s is the Euclidean arclength parameter, and $k = \langle \kappa, \nu \rangle = \langle \gamma_{ss}, \nu \rangle$. We choose ν by setting $\nu = (\tau_2, -\tau_1)$ where $\tau = \gamma_s$ is the tangent vector with direction induced by the given parametrisation. We call η_i *supporting curves* for the flow.

The length of γ is

$$L(\gamma) = \int_{-1}^1 |\gamma_u| du.$$

Another important quantity, in addition to the elastic energy E introduced earlier, is

$$(2) \quad A(\gamma) = -\frac{1}{2} \int_{-1}^1 \langle \gamma, \nu \rangle |\gamma_u| du,$$

which is the usual notion of area for closed plane curves. Here, A corresponds to the area of the star-shaped domain (with multiplicity) traced out by rays connecting the position vector γ and the origin.

Consider a one-parameter family of immersed curves $\gamma : [-1, 1] \times [0, T) \rightarrow \mathbb{R}^2$ satisfying the boundary conditions (1) and have normal speed given by F , that is

$$\partial_t \gamma = -F \nu.$$

The flows are:

(CD): normal velocity equal to $-\text{grad}_{H^{-1}}(L(\gamma))$, that is,

$$F = k_{ss};$$

(E): normal velocity equal to $-\text{grad}_{L^2}(E(\gamma))$, that is,

$$F = k_{ss} + \frac{1}{2} k^3;$$

The (free) boundary value problem that we wish to consider for these flows is the following:

$$(CD/E) \quad \begin{cases} (\partial_t \gamma)(u, t) = -(F\nu)(u, t) & \text{for all } (u, t) \in (-1, 1) \times (0, T) \\ \gamma(-1, t) \in \eta_1(\mathbb{R}); \quad \gamma(1, t) \in \eta_2(\mathbb{R}) & \text{for all } t \in [0, T) \\ \langle \nu, \nu_{\eta_1} \rangle(-1, t) = \langle \nu, \nu_{\eta_2} \rangle(1, t) = 0 & \text{for all } t \in [0, T) \\ k_s(-1, t) = k_s(1, t) = 0 & \text{for all } t \in [0, T). \end{cases}$$

Note that we do not prescribe the tangential movement in (CD/E). In the closed case, tangential movements leave the image invariant and correspond to reparametrisations in the domain. For the boundary case, this is no longer true and tangential movements typically correspond to stretching the image (if not periodic for example). We therefore have no freedom in choosing a tangential movement that will simplify analysis, as it will typically be forced upon us by the existence theory. Nevertheless tangential motion, as in the closed case, plays almost no role in the (global) analysis as all quantities that arise from the commutator relations (see Lemma 2.1) depend only on the normal component of the velocity.

The curve diffusion flow is the steepest descent gradient flow for length in H^{-1} . Since the velocity is a potential, signed enclosed area A in the case of closed curves is constant along the flow. This shows that the isoperimetric ratio is a scale-invariant monotone quantity for the flow, and this fact can be useful for analysis of solutions to the flow (see [62] for example). In the case of the boundary problems considered here, this is no longer true. Here it is difficult to find a useful notion of enclosed area. Indeed, this is a fundamental obstacle to smooth compactness, and can only be overcome in the case when the flow is already in its preferred topological class, that is, when we assume that $\omega = 0$ (see Remark 4 and Figure 3).

Local existence for (CD/E) can be proved by using the standard procedure of solving the flow in the class of graphs over the initial data, as in [56]. As we consider a Neumann problem, we may use a local adapted coordinate system similar to Stahl [56] which does not require a tangential component in the velocity of the flow. This can be continued until the solution leaves this class, at which point there is either some loss of regularity in $C^{4,\alpha}$, or the solution is simply no longer graphical over its initial state. The latter problem is a technicality, and can be resolved by writing the flow in a new coordinate system, as a graph over the solution at a later time. Now if there are uniform $C^{2,\alpha}$ -estimates, it is possible to use a standard contraction map argument to continue the solution. To the best of our knowledge the first to observe that only $C^{2,\alpha}$ is required were Ito-Kohsaka, with the map Φ constructed in [30, Proof of Theorem 3.1]. There they are working with (CD) however the additional term added by (E) does not cause any additional difficulty. Therefore by iterating the above procedure we find that the maximal time of existence is either infinity, or the $C^{2,\alpha}$ norm has blown up. In this paper, the most natural norms to control a-priori are L^2 in arc-length derivatives of curvature. The standard Sobolev inequality allows us to control the $C^{2,\alpha}$ norm by the length of the position vector $|\gamma|$ and the L^2 -norm of the first derivative of curvature. Note that it is not (without additional arguments) enough to bound only the length of the evolving curves. The statement below is specialised to our current situation, where the supporting curves are straight lines. We note that it is far from optimal.

Theorem 1.1 (Local existence). *Let $\eta_i : \mathbb{R} \rightarrow \mathbb{R}^2$ ($i = 1, 2$) be straight lines. Suppose $\gamma_0 : \mathbb{R} \rightarrow \mathbb{R}^2$ is a regular smooth curve satisfying the boundary conditions (1). Then there*

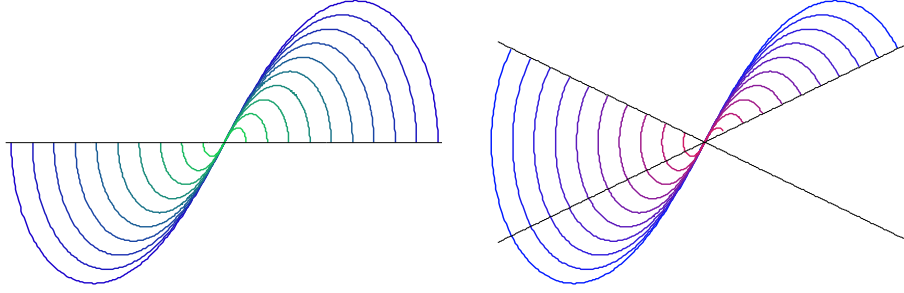


FIGURE 1. The curve diffusion flow with free boundary becoming singular in finite time. The evolution is homothetic.

exists a maximal $T \in (0, \infty]$ and a unique one-parameter family of regular immersed curves $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ satisfying $\gamma(u, 0) = \gamma_0(u)$ and (CD/E). Furthermore, if $T < \infty$, then there does not exist a constant C such that

$$(3) \quad \|\gamma\|_\infty + \|k_s\|_2 \leq C$$

for all $t \in [0, T)$.

Remark 1. If the flow is not supported on straight lines, then we require compatibility conditions to produce a solution. If the compatibility conditions are violated by the initial data, then we are still typically able to produce a flow, however convergence as $t \searrow 0$ will be limited by the degree to which the compatibility conditions are satisfied. One interesting investigation into this for the surface diffusion flow is [1], where the degree of incompatibility is finely studied in the context of the original motivation from Mullins [46].

1.1. Curve diffusion flow. In light of condition (3), global existence follows if we are able to uniformly bound the length of the position vector and the L^2 -norm of the derivative of curvature. The curve diffusion flow is the H^{-1} gradient flow of the length functional, with $L' = -\|k_s\|_2^2$. The length is uniformly controlled a-priori but this does not yield immediately an estimate for $\|\gamma\|_\infty$. It does make $\|k_s\|_2^2$ a natural energy for the flow, with an a-priori uniform estimate in $L^1([0, T))$ depending only on the length of the initial data. Despite this, there are shrinking self-similar solutions to the evolution equation (see Figure 1, which relies upon the lemniscate described in [19]) that are clearly singular in finite-time. Additionally, there is a conjecture due to Giga that implies finite-time singularities can occur from initially embedded data. For the situation with free boundary considered here, we expect that there exist a greater variety of such singularities.

Therefore global existence is not expected to hold generically. It is natural to hope however that in a suitable neighbourhood of minimisers for the energy, global existence and convergence to a minimiser holds. The only global minimisers are straight lines perpendicular to the supporting lines. Our main theorem confirms that these equilibria are stable, with neighbourhood given by the oscillation of curvature.

First let us define:

- Set e to be any vector such that all minimisers of length are translates of e .

- The constant ω and the average curvature are defined by

$$\int_{\gamma} k \, ds \Big|_{t=0} = 2\omega\pi,$$

$$\bar{k}(\gamma) = \frac{1}{L} \int_{\gamma} k \, ds.$$

Note that ω is not typically an integer (see Lemma 2.5).

- The oscillation of curvature and the isoperimetric ratio are defined as

$$K_{osc}(\gamma) = L \int_{\gamma} (k - \bar{k})^2 \, ds,$$

and

$$I(\gamma) = \frac{L^2(\gamma)}{4\omega\pi A(\gamma)}.$$

Theorem 1.2. *Suppose $|e| > 0$. Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD). Suppose γ_0 satisfies*

$$(4) \quad L(\gamma_0) \|k\|_2^2(\gamma_0) < \frac{\pi}{10}.$$

Then $\omega = 0$, the flow exists globally $T = \infty$, and $\gamma(\cdot, t)$ converges exponentially fast to a translate of e in the C^∞ topology.

Remark 2. The hypothesis of Theorem 1.2 implies that $\omega = 0$. To see this, we calculate at initial time

$$(2\omega\pi)^2 = \left(\int_{\gamma} k \, ds \right)^2 \leq L \int_{\gamma} k^2 \, ds < \frac{\pi}{10}$$

so

$$\omega^2 < \frac{\pi}{10} \frac{1}{4\pi^2} < \frac{1}{4}.$$

The boundary condition implies that ω is an integer multiple of $\frac{1}{2}$, and so must be zero. As ω is constant along the flow (see Lemma 2.5), it remains zero for all time.

Remark 3. Identifying which translate the solution converges to is a difficult open problem, similar to the problem of identifying the location of the final point singularity that planar curve shortening flow approaches (see [7]).

For closed curves, if the oscillation of curvature is initially small, then the flow exists for all time and converges exponentially fast to a standard circle. This is the main result of [62]. Also in [62] is an estimate of the *waiting time*: as the limit is a circle and convergence is smooth, there exists a T^* such that $k > 0$ for all $t > T^*$, that is, the flow is eventually convex. This is interesting in light of [28], that shows convexity is in general lost under the flow.

This is a symptom of the failure of the maximum principle for fourth-order differential operators. Another such symptom was identified by Elliott and Maier-Paape [20], that graphicality is typically lost in finite time. In our situation here, a natural ‘graph direction’ exists: the rotation of e by $\frac{\pi}{2}$. Let us denote this rotated vector by f . Indeed, analogously to the situation in [62], there exists a waiting time T^* such that for all $t > T^*$, we have

$$f[\gamma](x, t) := \langle \nu(x), f \rangle > 0, \quad \text{for all } x \in (-1, 1).$$

That is, the flow is eventually graphical. This leads us to the natural question:

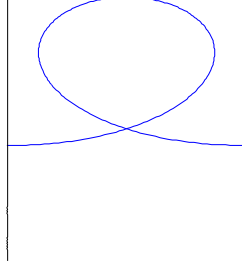


FIGURE 2. Sample initial data. This initial data has winding number 1.

Question. Suppose $|e| > 0$. Let $\gamma : (-1, 1) \times [0, \infty) \rightarrow \mathbb{R}^2$ be a solution to (CD) satisfying the assumptions of Theorem 1.2. Does there exist a $C = C(\gamma(\cdot, 0))$ depending only on the initial data such that

$$\mathcal{L}\{t \in [0, \infty) : f[\gamma](\cdot, t) \not\equiv 0\} \leq C(\gamma(\cdot, 0))$$

and for every $\varepsilon > 0$ there exists a flow γ_ε such that

$$\mathcal{L}\{t \in [0, \infty) : f[\gamma](\cdot, t) \not\equiv 0\} > C(\gamma_\varepsilon(\cdot, 0)) - \varepsilon?$$

In the above we have used \mathcal{L} to denote Lebesgue measure.

Remark 4. Finite-time singularities for the curve diffusion flow with closed data remain difficult to penetrate. Although there are natural Lyapunov functionals for the flow, these do not seem to yield classification results for blowups of singularities. Indeed, it is still unknown if solutions in symmetric perturbation classes near non-trivial shrinkers (such as the figure-8 solution discussed in [19]) converge modulo rescaling to the shrinker. As mentioned, we can also understand this self-similar solution in the free boundary setting (see Figure 1). It seems likely that the free boundary setting will be useful when studying perturbations of the figure-8.

In the free boundary setting, finite-time singularities are more common, and global analysis of the flow can be quite problematic even in a small data regime. For example, the exterior problem, where the flow is supported on parallel lines but with winding number $\omega \neq 0$ (see Figure 3) is in a class of curves whose members all have non-constant curvature. There is no equilibrium in that setting satisfying the boundary conditions. Nevertheless, by adjusting the aperture width $|e|$, it is simple to see that one may make the oscillation of curvature arbitrarily small.

There is an interesting technical point here. Some of the estimates used to prove Theorem 1.2 are close to optimal: using initial smallness of the oscillation of curvature, we may use the method of proof from Lemma 3.10 to find that curvature is well-controlled in L^2 if we can control the length difference $L(\gamma_t) - L(\gamma_0)$. If the supporting lines are skew, this follows by using an isoperimetric-type argument. For parallel lines this doesn't work. If $\omega = 0$ then the problematic term is absent, however for $\omega \neq 0$, the term needs to be estimated. An easy condition controlling this term is that $L(\gamma_0) = |e| + \delta$, where $|e|$ is the length of the straight line connecting each of the parallel lines. If it were possible to choose $\delta < K_0$, where K_0 is larger than the initial oscillation of curvature and smaller

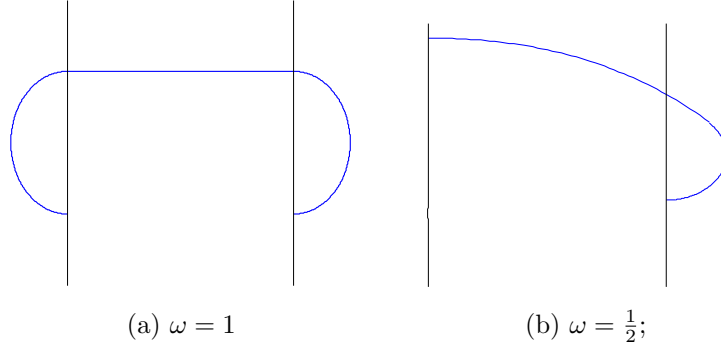


FIGURE 3. Sample initial data for the exterior problem.

than K^* from Lemma 3.10, then a stability result would follow. These requirements are in competition with one another: although the oscillation of curvature is scale-invariant, decreasing δ beyond a certain critical level necessitates an increase in the oscillation of curvature. Indeed, the fact that there is no equilibrium in the class of curves satisfying the boundary conditions for $\omega \neq 0$ proves that it is not possible to make this choice. As a corollary of this, we conclude the following lower bound for the oscillation of curvature in the exterior problem.

Corollary 1.3. *Suppose $|e| > 0$. Let $\gamma : (-1, 1) \rightarrow \mathbb{R}^2$ be an immersed curve satisfying the boundary conditions of the exterior problem: $\eta_i : \mathbb{R} \rightarrow \mathbb{R}^2$ are parallel straight lines, the origin lies in the interior of η_i ¹, γ meets η_i at right angles, with $k_s(\pm 1) = 0$, and at least one of the tangent vectors at the boundary τ_i points away from the interior of η_i .*

Then

$$K_{osc}(\gamma) + 8\pi^2 \log \left(\frac{L(\gamma)}{|e|} \right) \geq \frac{12\pi^2\omega^2 + \pi - 2\omega\pi\sqrt{6\pi(6\pi\omega^2 + 1)}}{3}.$$

Concerning the global behaviour of this flow, we make the following conjecture.

Conjecture. *Suppose $|e| > 0$. Let $\gamma_0 : (-1, 1) \rightarrow \mathbb{R}^2$ be an immersed curve satisfying the boundary conditions of the exterior problem, as in Corollary 1.3. The curve diffusion flow with free boundary $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ with γ_0 as initial data exists for at most finite time, and $\gamma(\cdot, t)$ approaches a multiply-covered straight line in the C^0 topology and not in C^k for any $k \geq 1$.*

1.2. Elastic flow. We finish by giving a surprising global result on the vanilla elastic flow. As noted earlier, despite $\|k\|_2^2$ being uniformly bounded and non-increasing along an elastic flow, compactness is not expected in general due to the norm $\|\gamma\|_\infty$ typically growing without bound. In order to obtain compactness, a restriction on length is usually imposed.

This makes global results on the vanilla elastic flow quite rare. For the flow supported on parallel lines, we are able to obtain a result of this kind, if the initial oscillation of curvature is not bigger than π . This can be thought of as a stability result for straight lines,

¹The interior is the region between the two parallel lines.

as the boundary condition can, via reflection, be understood as imposing a *cocompactness condition* on the flow.

Theorem 1.4. *Suppose $|e| > 0$. Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E). Assume that*

$$(5) \quad L(\gamma_0) \int_{\gamma} k^2 ds \Big|_{t=0} \leq \pi.$$

Then the flow exists globally $T = \infty$ and $\gamma(\cdot, t)$ converges exponentially fast to a translate of e in the C^∞ topology.

Remark 5. As with (CD), it is unknown how to determine, from the initial data, which straight line the flow will converge to.

Sharpness of the given condition is unknown, however, we do not expect it to be sharp. Based on the winding number calculation in Lemma 3.2 and numerical evidence, we make the following conjecture.

Conjecture. *Theorem 1.4 holds with (5) replaced by*

$$(6) \quad L(\gamma_0) \int_{\gamma} k^2 ds \Big|_{t=0} \leq \pi^2.$$

The argument for including equality in (6) above is as follows. It is possible to construct, for any $\delta > 0$, a curve satisfying the boundary conditions with

$$\omega = \frac{1}{2} \quad \text{and} \quad K_{osc} = \pi^2 + \delta.$$

Clearly such curves can not smoothly converge to a straight line; in fact, numerical evidence suggests that (unlike (CD) flow) such curves expand indefinitely and do not display any compactness property. In particular, length is no longer controlled a-priori.

However the limit as $\delta \searrow 0$ has $\omega = 0$ and this does not seem to be avoidable. This is why we conjecture that the sharp energy level that allows compactness and smooth convergence is π^2 , with π^2 included.

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2. EVOLUTION EQUATIONS AND STANDARD INEQUALITIES

Changes in the Euclidean geometry of the evolving curves can be understood via the noncommutativity of the Euclidean arc-length and time derivatives. In this section and throughout the rest of the paper we reparametrise the evolving family of curves by arc-length, with arc-length parameter s .

Lemma 2.1. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD/E) given by Theorem 1.1. Then*

$$[\partial_t, \partial_s] = \partial_t \partial_s - \partial_s \partial_t = -kF \partial_s.$$

Proof. We compute

$$\begin{aligned} \partial_t |\gamma_u|^2 &= 2|\gamma_u| |\gamma_u|_t \\ &= 2 \langle \gamma_u, \gamma_{tu} \rangle = 2 \langle \gamma_u, (-F\nu)_u \rangle \\ &= -2F \langle \gamma_u, \nu_u \rangle \\ &= 2F \langle \gamma_u, \gamma_u \rangle \\ &= 2(kF) |\gamma_u|^2; \end{aligned}$$

so

$$(7) \quad \partial_t |\gamma_u| = (kF) |\gamma_u|$$

and

$$\partial_t \partial_s - \partial_s \partial_t = \partial_t \left(\frac{1}{|\gamma_u|} \right) \partial_u = \frac{-kF}{|\gamma_u|} \partial_u = -kF \partial_s.$$

□

The commutator relation allow us to quickly calculate the evolution of the tangent, normal, curvature, and derivative of curvature vectors.

Lemma 2.2. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD/E) given by Theorem 1.1. The following evolution equations hold:*

$$\begin{aligned} \tau_t &= -F_s \nu \\ \nu_t &= F_s \tau \\ k_t &= -F_{ss} - Fk^2 \\ k_{st} &= -F_{s3} - F_s k^2 - 3Fk_s k. \end{aligned}$$

Proof. Lemma 2.1 is used repeatedly in this proof. Let us begin with

$$\begin{aligned} \tau_t &= \gamma_{st} = \gamma_{ts} - kF \gamma_s \\ &= -(F\nu)_s - kF \tau \\ &= -F_s \nu + (kF - kF) \tau \\ &= -F_s \nu. \end{aligned}$$

Since $\langle \tau, \nu \rangle = 0$, this implies directly

$$\nu_t = -\langle \nu, -F_s \nu \rangle \tau = F_s \tau.$$

Similarly,

$$\begin{aligned} \kappa_t &= \gamma_{sst} = \gamma_{sts} - kF \gamma_{ss} \\ &= -(F_s \nu)_s - kF \kappa \\ &= -(F_{ss} + k^2 F) \nu + kF_s \tau. \end{aligned}$$

The tangential movement of the curvature vector here can be understood as rotation, whereas the normal velocity is a dilation. Indeed, the scalar curvature $k = \langle \nu, \kappa \rangle$ evolves by

$$\begin{aligned} k_t &= \langle F_{ss}\tau, \kappa \rangle + \langle -(F_{ss} + k^2 F)\nu + (kF_s)\tau, \nu \rangle \\ &= -F_{ss} - Fk^2. \end{aligned}$$

For $\kappa_s = k_s\nu - k^2\tau$ we proceed as before:

$$\begin{aligned} \kappa_{st} &= \kappa_{ts} - kF\kappa_s \\ &= [(-F_{ss} - k^2 F)\nu + kF_s\tau]_s - kF\kappa_s \\ &= (-F_{s^3} - 2kk_sF - k^2F_s + k^2F_s - kk_sF)\nu + X\tau \\ &= (-F_{s^3} - 3kk_sF)\nu + X\tau. \end{aligned}$$

Since $k_s = \langle \kappa_s, \nu \rangle$, this implies

$$\begin{aligned} k_{st} &= \langle F_s\tau, k_s\nu - k^2\tau \rangle \\ &\quad + \langle (-F_{s^3} - 3kk_sF)\nu + (\dots)\tau, \nu \rangle \\ &= -k^2F_s - F_{s^3} - 3kk_sF \\ &= -F_{s^3} - F_s k^2 - 3Fk_s k. \end{aligned}$$

□

Remark 6. We note that the above evolution equation can be obtained without requiring explicit calculation of κ_{st} by working directly with the scalar quantites:

$$\begin{aligned} k_{st} &= k_{ts} - Fk_s k \\ &= (-F_{ss} - Fk^2)_s - Fk_s k \\ &= -F_{s^3} - F_s k^2 - 3Fk_s k. \end{aligned}$$

We included the method using the evolution of the vector κ_s as we feel the quantity κ_{st} , and in particular the nontrivial cancellations involved in moving from $\kappa_s \mapsto k_s$ are also of interest.

Induction on the commutator relations yields the following generic formula for the evolution of the l -th derivative of curvature. Similar formulae were derived in [18, Lemma 2.3].

Lemma 2.3. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD/E) given by Theorem 1.1. The evolution of the l -th derivative of curvature*

(CD) *along the curve diffusion flow is given by*

$$\partial_t k_{s^l} = -k_{s^{(l+4)}} + \sum_{q+r+u=l} c_{qru} k_{s^{(q+2)}} k_{s^r} k_{s^u},$$

for constants $c_{qru} \in \mathbb{R}$ with $q, r, u \geq 0$;

(E) *along the elastic flow is given by*

$$\partial_t k_{s^l} = -k_{s^{(l+4)}} + \sum_{q+r+u=l} c_{qru} k_{s^{(q+2)}} k_{s^r} k_{s^u} + \sum_{q+r+u+v+w=l} c_{qruvw} k_{s^q} k_{s^r} k_{s^u} k_{s^v} k_{s^w},$$

for constants $c_{qru}, c_{qruvw} \in \mathbb{R}$ with $q, r, u, v, w \geq 0$.

Equation (7) gives the evolution of the length element, and thus, of the change in length of the evolving curves.

Lemma 2.4. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD/E) given by Theorem 1.1. Then*

$$L'(\gamma(\cdot, t)) = \int Fk \, ds.$$

In particular for (CD) flow we have

$$L'(\gamma(\cdot, t)) = - \int k_s^2 \, ds.$$

Proof. Using (7) we compute

$$\frac{d}{dt} \int_{-1}^1 |\gamma_u| \, du = \int_{-1}^1 Fk |\gamma_u| \, du.$$

If $F = k_{ss}$ then

$$L'(\gamma(\cdot, t)) = - \int k_s^2 \, ds,$$

as required. Note that we used the boundary conditions in the last step. \square

Note that the elastic flow tends to increase length and for the constrained elastic flow although we have $L(\gamma(\cdot, t)) \leq L(\gamma(\cdot, 0))$, we do not have monotonicity of the length in general.

There exists an $\omega \in \mathbb{R}$ satisfying

$$(8) \quad \int_{\gamma} k \, ds \Big|_{t=0} = 2\omega\pi.$$

In the case where the solution is a family of closed curves, ω is the winding number of $\gamma(\cdot, 0)$. In the cases we consider here, we continue to call ω the winding number, although it is no longer guaranteed to be an integer. The lemma below shows that, as expected, it is constant along the flow. Note that it requires the free boundary condition and the flatness of the support lines η_i .

Lemma 2.5. *Suppose $|e| > 0$. Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD/E) given by Theorem 1.1. Set*

$$\int_{\gamma} k \, ds \Big|_{t=0} = 2\omega\pi.$$

Then

$$\int_{\gamma} k \, ds = 2\omega\pi.$$

In particular, for (CD) flow the average of the curvature \bar{k} increases in absolute value with velocity

$$\frac{d}{dt} \bar{k} = \frac{2\omega\pi}{L^2} \|k_s\|_2^2.$$

Proof. We set the origin to be any point on the line equidistant from the two parallel lines η_i . Recall that e is a vector perpendicular to η_i and has length equal to the (minimum) distance between them. The Neumann condition is equivalent to

$$\langle \nu(\pm 1, t), e \rangle = 0.$$

Differentiating this in time yields

$$F_s(\pm 1, t) \langle \tau(\pm 1, t), e \rangle = \pm |e| F_s(\pm 1, t) = 0.$$

Now $|e| \neq 0$ so we must have that

$$(9) \quad F_s(\pm 1, t) = 0.$$

We compute

$$\frac{d}{dt} \int_{\gamma} k ds = - \int_{\gamma} F_{ss} + Fk^2 - Fk^2 ds = - \int_{\gamma} F_{ss} ds = -F_s|_{\{0, L\}} = 0.$$

This completes the proof. \square

Differentiating the boundary conditions can be taken quite far, as the following lemma shows.

Lemma 2.6. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD/E) given by Theorem 1.1. For all $p \in \mathbb{N}$*

$$k_{s^{(2p+1)}} \Big|_{\{0, L\}} = 0.$$

Proof. Differentiating the no-flux condition $k_s(\pm 1, t) = 0$ we find

$$(10) \quad (-F_{s^3} - F_s k^2 - 3Fk_s k)(\pm 1, t) = 0.$$

Substituting the no-flux condition and (9) into (10) we find

$$(11) \quad F_{s^3}(\pm 1, t) = 0.$$

For the (CD/E) flows these imply together with an induction argument that

$$(12) \quad k_{s^{(2p-1)}}(\pm 1, t) = 0$$

for $p = 1, 2, 3$. For clarity we calculate in each case separately.

(CD) flow. For curve diffusion flow the claim (12) is immediate.

(E) flow. For the elastic flow we have at the boundary

$$0 = F_s = k_{s^3} + \frac{3}{2}k^2 k_s = k_{s^3}$$

and

$$\begin{aligned} 0 &= F_{s^3} = k_{s^5} + \frac{3}{2}(k^2 k_s)_{ss} \\ &= k_{s^5} + \frac{3}{2}(2k k_s^2 + k^2 k_{ss})_s \\ &= k_{s^5} + \frac{3}{2}(2k_s^3 + 6k k_s k_{ss} + k^2 k_{s^3}) \\ &= k_{s^5}. \end{aligned}$$

We conclude again (12).

Let us give the induction argument. We assume that for all $p \in 1, \dots, n$,

$$(13) \quad k_{s^{(2p-1)}}(\pm 1, t) = 0.$$

The evolution of the l -th derivative of curvature is given by

$$\partial_t k_{s^l} = -k_{s^{(l+4)}} + \sum_{q+r+u=l} c_{qru} k_{s^{(q+2)}} k_{s^r} k_{s^u} + \sum_{q+r+u+v+w=l} c_{qruvw} k_{s^q} k_{s^r} k_{s^u} k_{s^v} k_{s^w},$$

for constants $\hat{c}_{gru}, c_{gru}, c_{gruvw} \in \mathbb{R}$ with $q, r, u, v, w \geq 0$. Note that for curve diffusion flow $c_{gruvw} = 0$.

The inductive hypothesis implies that, for l odd and less than or equal to $2n - 1$, the derivative k_{s^l} vanishes on the boundary. Let's take $l = 2n - 3$. Then we have (evaluating at the boundary)

$$\begin{aligned} k_{s(2n+1)} = & - \sum_{2q_1+2r_1+2u_1=2n-3} k_{s(2q_1+2)} * k_{s(2r_1)} * k_{s(2u_1)} \\ & - \sum_{2q_2+2r_2+2u_2+2v_2+2w_2=2n-3} k_{s(2q_2)} * k_{s(2r_2)} * k_{s(2u_2)} * k_{s(2v_2)} * k_{s(2w_2)}. \end{aligned}$$

In the above equation we have used $*$ to denote a linear combination of terms with absolute coefficient. Using the hypothesis (13), we have removed all terms with an odd number of derivatives of k . Each sum is therefore taken over all q_i, r_i, u_i, v_i, w_i such that $2q_i + 2r_i + 2u_i = 2p - 3$ or $2q_i + 2r_i + 2u_i + 2v_i + 2w_i = 2p - 3$, which is the empty set. We conclude

$$k_{s(2n+1)} = 0,$$

equivalent to (12), as required. \square

We will need the following Sobolev-Poincaré-Wirtinger inequalities. The statements below are in the arc-length parametrisation, but we note that they continue to hold under any change of measure.

Lemma 2.7. *Suppose $f : [0, L] \rightarrow \mathbb{R}$, $L > 0$, is absolutely continuous and $\int_0^L f ds = 0$. Then*

$$\int_0^L f^2 ds \leq \frac{L^2}{\pi^2} \int_0^L f_s^2 ds.$$

Proof. Form the new function $g : [-L, L] \rightarrow \mathbb{R}$ defined by $g(x) = f(x)$ for $x > 0$ and $g(x) = f(-x)$ for $x \leq 0$. Then g is periodic, since $g(-L) = g(L)$, and furthermore

$$\int_{-L}^L g ds = 2 \int_0^L f ds = 0.$$

Applying the standard Poincaré inequality to g we find

$$\int_{-L}^L g^2 ds \leq \frac{L^2}{\pi^2} \int_{-L}^L g_s^2 ds.$$

This implies

$$2 \int_0^L f^2 ds \leq \frac{2L^2}{\pi^2} \int_0^L f_s^2 ds,$$

as required. \square

Lemma 2.8. *Suppose $f : [0, L] \rightarrow \mathbb{R}$, $L > 0$, is absolutely continuous and $f(0) = f(L) = 0$. Then*

$$\int_0^L f^2 ds \leq \frac{L^2}{\pi^2} \int_0^L f_s^2 ds.$$

Proof. Define $g(s) = f(s)$ for $s \in [0, L]$ and $g(s) = -f(-s)$ for $s \in [-L, 0)$. Then g is a continuous odd periodic function. Since $\int g ds = 0$, the standard Poincaré inequality implies

$$2 \int_0^L f^2 ds = \int_{-L}^L g^2 ds \leq \frac{4L^2}{4\pi^2} \int_{-L}^L g_s^2 ds = \frac{2L^2}{\pi^2} \int_0^L f_s^2 ds.$$

□

Corollary 2.9. *Under the assumptions of Lemma 2.8*

$$\|f\|_\infty^2 \leq \frac{L}{\pi} \|f'\|_2^2.$$

Under the assumptions of Lemma 2.7

$$\|f\|_\infty^2 \leq \frac{2L}{\pi} \|f'\|_2^2.$$

Proof. First assume $f : [0, L] \rightarrow \mathbb{R}$, $L > 0$, is absolutely continuous and $f(0) = f(L) = 0$. Periodicity implies

$$f^2(s) = \int_0^s f f' ds - \int_s^L f f' ds.$$

Therefore

$$f^2(s) \leq \int_0^L |f f'| ds.$$

Now Hölder's inequality and Lemma 2.8 above implies

$$\|f\|_\infty^2 \leq \|f\|_2 \|f'\|_2 \leq \frac{L}{\pi} \|f'\|_2^2,$$

as required.

It remains to prove the lemma in the case where $f : [0, L] \rightarrow \mathbb{R}$, $L > 0$, is absolutely continuous and $\int_0^L f ds = 0$. In this case there exists a point $p \in [0, L]$ where $f(p) = 0$. Absolute continuity implies

$$f^2(s) = 2 \int_p^s f f' ds.$$

Therefore

$$\|f\|_\infty^2 \leq 2 \|f\|_2 \|f'\|_2 \leq \frac{2L}{\pi} \|f'\|_2^2.$$

□

We now compute the evolution of K_{osc} for (CD) flow.

Lemma 2.10. *Let $\gamma : (-1, 1) \times [0, T] \rightarrow \mathbb{R}^2$ be a solution to (CD) given by Theorem 1.1. Then*

$$\begin{aligned} & \frac{d}{dt} K_{osc} + K_{osc} \frac{\|k_s\|_2^2}{L} + 2L \|k_{ss}\|_2^2 \\ &= 3L \int_\gamma (k - \bar{k})^2 k_s^2 ds + 6\bar{k}L \int_\gamma (k - \bar{k}) k_s^2 ds + 2\bar{k}^2 L \|k_s\|_2^2. \end{aligned}$$

Proof. First, note that

$$2L \int_{\gamma} (k - \bar{k})(-k_{s^4}) ds = -2L \|k_{ss}\|_2^2 + 2L [k_s k_{ss}]_{\{0,L\}} - 2L [(k - \bar{k})k_{s^3}]_{\{0,L\}}.$$

From (9) in the proof of Lemma 2.5, the last term vanishes. The no-flux boundary condition means that the second term vanishes. Therefore,

$$2L \int_{\gamma} (k - \bar{k})(-k_{s^4}) ds = -2L \|k_{ss}\|_2^2.$$

The no-flux boundary condition also means that

$$2L \int_{\gamma} (k - \bar{k})(-k^2 k_{ss}) ds = 2L \int_{\gamma} k^2 k_s^2 ds + 4L \int_{\gamma} (k - \bar{k}) k k_s^2 ds$$

and

$$L \int_{\gamma} k k_{ss} (k - \bar{k})^2 ds = -L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds - 2L \int_{\gamma} k (k - \bar{k}) k_s^2 ds.$$

Using these identities, we compute

$$\begin{aligned} \frac{d}{dt} K_{osc} &= - \int_{\gamma} k_s^2 ds \int_{\gamma} (k - \bar{k})^2 ds + 2L \int_{\gamma} (k - \bar{k})(-k_{s^4} - k^2 k_{ss}) ds \\ &\quad + L \int_{\gamma} k k_{ss} (k - \bar{k})^2 ds \\ &= -K_{osc} \frac{\|k_s\|_2^2}{L} - 2L \|k_{ss}\|_2^2 + 2L \int_{\gamma} k^2 k_s^2 ds + 4L \int_{\gamma} k (k - \bar{k}) k_s^2 ds \\ &\quad - L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds - 2L \int_{\gamma} k (k - \bar{k}) k_s^2 ds \\ &= -K_{osc} \frac{\|k_s\|_2^2}{L} - 2L \|k_{ss}\|_2^2 + 4L \int_{\gamma} k^2 k_s^2 ds - 2\bar{k}L \int_{\gamma} k k_s^2 ds \\ &\quad - L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds. \end{aligned}$$

Rearranging, we have

$$\begin{aligned} \frac{d}{dt} K_{osc} + K_{osc} \frac{\|k_s\|_2^2}{L} + 2L \|k_{ss}\|_2^2 &= 4L \int_{\gamma} k^2 k_s^2 ds - 2\bar{k}L \int_{\gamma} (k - \bar{k}) k_s^2 ds - 2\bar{k}^2 L \int_{\gamma} k_s^2 ds - L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds \\ &= 3L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds + 6\bar{k}L \int_{\gamma} (k - \bar{k}) k_s^2 ds + 2\bar{k}^2 L \int_{\gamma} k_s^2 ds. \end{aligned}$$

This proves the lemma. \square

3. CURVATURE ESTIMATES IN L^2 AND GLOBAL ANALYSIS

In this section we describe how the evolution equations can be used to prove a-priori curvature estimates, and then how these estimates imply global existence and convergence.

A fundamental observation is that for the curve diffusion flow Lemma 2.4 and Lemma 2.7 together imply $K_{osc} \in L^1([0, T])$.

Lemma 3.1. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD) given by Theorem 1.1. Then $\|K_{osc}\|_1 < L^4(0)/4\pi^2$.*

Proof. Applying Lemma 2.7 with $f = k - \bar{k}$ and recalling Lemma 2.4 we have

$$K_{osc} \leq \frac{L^3}{\pi^2} \|k_s\|_2^2 = -\frac{1}{4\pi^2} \frac{d}{dt} L^4,$$

so

$$\int_0^t K_{osc} d\tau \leq \frac{L^4(\gamma_0)}{4\pi^2} - \frac{L^4(\gamma_t)}{4\pi^2}. \quad \square$$

The above lemma holds regardless of initial data, in particular regardless of the winding number, indicating that the quantity K_{osc} is a natural energy for the curve diffusion flow with free boundary, just as it is for the curve diffusion flow of closed curves [62].

Remark 7. A similar argument as above also shows that $\|k_s\|_2^2 \in L^1([0, T))$ with the estimate $\|\|k_s\|_2^2\|_1 \leq L(0)$, suggesting that $\|k_s\|_2^2$ is another well-behaved quantity under the flow.

For the elastic flow, the natural limit of the flow is a free elastica, which means that we do *not* expect K_{osc} to be a natural energy for the flow. We nevertheless retain a-priori control of $\|k\|_2^2$ for (E).

For the elastic flow supported on parallel lines, something interesting occurs: the flow converges to a zero of K_{osc} when K_{osc} is initially small. We begin by proving that there exists a critical energy level E such that while the oscillation of curvature is below E length is monotone.

Lemma 3.2. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that for all $t \in [0, T)$*

$$(14) \quad L \int_{\gamma} k^2 ds \leq \pi.$$

Then $\omega = 0$ and

$$L(\gamma(\cdot, t)) \leq L(\gamma(\cdot, 0)), \quad \text{for all } t \in [0, T).$$

Proof. Let us first show that $\omega = 0$. Note that Lemma 2.5 implies that ω is constant along the flow, and so it suffices to show that $\omega = 0$ at initial time. A calculation analogous to that in Remark 2 shows that

$$\omega \leq \frac{1}{2\sqrt{\pi}}.$$

As ω is an integer multiple of $\frac{1}{2}$, it must be zero.

Therefore by Lemma 2.9, we have

$$\|k\|_{\infty}^2 \leq \frac{2L}{\pi} \|k_s\|_2^2.$$

Using this we calculate

$$\begin{aligned}
L'(\gamma(\cdot, t)) &= \int_{\gamma} k k_{ss} + \frac{1}{2} k^4 ds \\
&= -\|k_s\|_2^2 + \frac{1}{2} \|k\|_4^4 \\
&\leq -\|k_s\|_2^2 + \frac{1}{2} \|k\|_{\infty}^2 \|k\|_2^2 \\
&\leq -\|k_s\|_2^2 \left(1 - \frac{L}{\pi} \|k\|_2^2\right) \\
&\leq 0.
\end{aligned}$$

The claim follows. \square

We note that the definition of the elastic flow implies $\|k\|_2^2$ is monotone decreasing, and so with Lemma 3.2 the product is also decreasing. This product, in the setting of $\omega = 0$, is K_{osc} , the oscillation of curvature.

Lemma 3.3. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that for $K \in \mathbb{R}$ we have*

$$(15) \quad L(\gamma_0) \int_{\gamma} k^2 ds \Big|_{t=0} \leq K \leq \pi.$$

Then for all $t \in [0, T)$

$$K_{osc}(\gamma(\cdot, t)) \leq K_{osc}(\gamma_0) = K.$$

Proof. By the calculation in the proof of Lemma 3.2 we have

$$L'(\gamma_0) \leq 0.$$

We calculate

$$\left(\frac{d}{dt} K_{osc}\right)(\gamma(\cdot, 0)) = L'(\gamma_0) \int_{\gamma_0} k^2 ds - 2L(\gamma_0) \int_{\gamma_0} \left|k_{ss} + \frac{1}{2}k^3\right|^2 ds \leq 0.$$

Therefore both L and K_{osc} are decreasing while $K_{osc} \leq K \leq \pi$. Since this is true at initial time, it remains true, and we conclude the estimate. \square

In fact we can obtain much more, bounding also $\|k_s\|_2^2$ a-priori.

Lemma 3.4. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that*

$$(16) \quad L(\gamma_0) \int_{\gamma} k^2 ds \Big|_{t=0} \leq \frac{4\pi}{7}.$$

Then for all $t \in [0, T)$

$$\int_{\gamma} k_s^2 ds \leq \frac{3L_0^2 K_1}{K_1 t + 3L_0^2}$$

where $L_0 = L(\gamma(\cdot, 0))$ and

$$K_1 = \int_{\gamma} k_s^2 ds \Big|_{t=0}.$$

Proof. We calculate

$$\begin{aligned}
\frac{d}{dt} \int_{\gamma} k_s^2 ds &= \int_{\gamma} k_s (-2F_{s^3} - 2F_s k^2 - 6F k_s k + k_s k F) ds \\
&= \int_{\gamma} -2k_{s^3} F_s - 2k_s k^2 F_s - 5k_s^2 k F ds \\
&= - \int_{\gamma} k_{s^3} (2k_{s^3} + 3k^2 k_s) ds \\
&\quad + \int_{\gamma} (k_s k^2)_s (2k_{s^3} + k^3) ds \\
&\quad - \frac{5}{2} \int_{\gamma} k_s^2 k (2k_{s^3} + k^3) ds \\
&= -2\|k_{s^3}\|_2^2 + 3 \int_{\gamma} k_{s^3} (k^2 k_{s^3} + 2k k_s^2) ds \\
&\quad - 2 \int_{\gamma} (k_s k^2) (k_{s^3}) ds - \int_{\gamma} k_s^2 k^4 ds \\
&\quad - \frac{5}{3} \int_{\gamma} (k_s^3)_s k ds - \frac{5}{2} \int_{\gamma} k_s^2 k^4 ds \\
&= -2\|k_{s^3}\|_2^2 + 3 \int_{\gamma} k^2 k_{s^3}^2 ds - \frac{1}{3} \int_{\gamma} k_s^4 ds \\
&\quad - 2 \int_{\gamma} (k_s k^2) (k_{s^3}) ds - \frac{7}{2} \int_{\gamma} k_s^2 k^4 ds.
\end{aligned}$$

Estimating with Young's inequality we find

$$\begin{aligned}
\frac{d}{dt} \int_{\gamma} k_s^2 ds &\leq -\left(2 - \frac{2}{7}\right) \|k_{s^3}\|_2^2 + 3 \int_{\gamma} k^2 k_{s^3}^2 ds - \frac{1}{3} \int_{\gamma} k_s^4 ds \\
&\leq -\frac{12}{7} \|k_{s^3}\|_2^2 + 3 \|k_{s^3}\|_{\infty}^2 \int_{\gamma} k^2 ds - \frac{1}{3} \int_{\gamma} k_s^4 ds \\
&\leq -\frac{12}{7} \|k_{s^3}\|_2^2 + \frac{3}{\pi} K_{osc} \|k_{s^3}\|_2^2 - \frac{1}{3L^2} \|k_s\|_2^4 \\
&\leq -\left(\frac{12}{7} - \frac{3}{\pi} K_{osc}\right) \|k_{s^3}\|_2^2 - \frac{1}{3L_0^2} \|k_s\|_2^4.
\end{aligned}$$

Now observe that if $K_{osc} \leq \varepsilon \leq \pi$ initially then this is by Lemma 3.3 preserved. Using this with $\varepsilon = \frac{\pi}{3} \frac{12}{7} = \frac{4\pi}{7}$, we obtain for $\phi = \|k_s\|_2^2$, $C = \frac{1}{3L_0^2}$,

$$\phi' \leq -C\phi^2$$

which implies

$$\phi(t) \leq \frac{\phi(0)}{C\phi(0)t + 1}$$

as required. \square

With Lemma 3.4 in hand, global existence and convergence to a straight line follows.

However, the hypothesis (16) is stronger than we would like; it is significantly worse than assuming simply $K_{osc} \leq \pi$, which is what is required for our a-priori length estimate.

Furthermore, blowup analysis indicates that we may bypass the estimate entirely. Assuming that $\|k_s\|_2^2(t_j) \rightarrow \infty$ for a sequence of times $t_j \rightarrow T < \infty$ we consider a sequence of rescalings $\gamma_j(u, t) = r_j \gamma(u, t_j + r_j^{-4}t)$. The sequence $\{r_j\}$ is chosen such that $\|k_s\|_{2, B_1}^2 = 1$. By a standard compactness theorem and local estimates for the flow we extract a limit γ_∞ which is an entire elastic flow. Now this limit has $\|k_s\|_2^2 \geq 1$ by construction, however by scale invariance and monotonicity of K_{osc} it is an elastica. Elasticae can exist at a variety of energy levels, however if K_{osc} is finite for an entire curve, then $\int k^2 ds = 0$, and we have a straight line, contradicting the concentration of $\int k_s^2 ds$.

This is not a proof, primarily because K_{osc} does not converge to anything and makes no sense on an entire curve as the length is infinite. However it does clearly suggest that we can do better than (16).

We present here a proof. First, the interpolation approach of Dziuk-Kuvert-Schätzle [18] allows us to establish global existence under the ideal hypothesis that $K_{osc} \leq \pi$.

Lemma 3.5. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that*

$$L(\gamma_0) \int_\gamma k^2 ds \Big|_{t=0} \leq \pi.$$

Then there exists a universal constant $C \in (0, \infty)$ such that for all $t \in [0, T)$

$$\int_\gamma k_s^2 ds \leq \int_\gamma k_s^2 ds \Big|_{t=0} + C.$$

Proof. The previous proof established that

$$\frac{d}{dt} \int_\gamma k_s^2 ds \leq -\frac{12}{7} \|k_{s^3}\|_2^2 + 3 \int_\gamma k^2 k_{ss}^2 ds - \frac{1}{3} \int_\gamma k_s^4 ds.$$

An interpolation inequality from Dziuk-Kuvert-Schätzle [18, Proposition 2.5] (see [14, Appendix C] for details, which additionally deals with non-closed curves), we find implies that there exists a universal constant C such that

$$3 \int_\gamma k^2 k_{ss}^2 ds \leq \frac{5}{7} \int_\gamma k_{s^3}^2 ds + C \|k\|_2^{14}.$$

Therefore

$$\begin{aligned} \frac{d}{dt} \int_\gamma k_s^2 ds &\leq -\|k_{s^3}\|_2^2 - \frac{1}{3} \int_\gamma k_s^4 ds + C \|k\|_2^{14} \\ &\leq -\frac{1}{3L} \left(\int_\gamma k_s^2 ds \right)^2 + C \pi^7 |e|^{-7}, \end{aligned}$$

so that, with $\phi(t) = \|k_s\|_2^2(t)$

$$\phi'(t) + c\phi^2(t) \leq C.$$

The result follows. \square

Corollary 3.6. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that*

$$L(\gamma_0) \int_\gamma k^2 ds \Big|_{t=0} \leq \pi.$$

Then $T = \infty$.

Proof. If $T < \infty$ then either $\|\gamma\|_\infty$ or $\|k_s\|_2$ is not bounded on $[0, T)$. By Lemma 3.5 we know that $\|k_s\|_2^2$ is uniformly bounded. This in turn implies that there exists a constant universal on $[0, T]$ for any $T > 0$ such that

$$L\|k\|_\infty \leq C.$$

For the position vector we calculate

$$\begin{aligned} \frac{d}{dt} \int_\gamma |\gamma|^2 ds &= \int_\gamma \langle \gamma, 2F\nu + kF\gamma \rangle ds \\ &\leq \int_\gamma k_{ss}(2\langle \gamma, \nu \rangle + k|\gamma|^2) ds + 2\|k\|_\infty^3 \int_\gamma |\gamma| ds + \|k\|_\infty^4 \int_\gamma |\gamma|^2 ds \\ &\leq \int_\gamma k_{ss}(2\langle \gamma, \nu \rangle + k|\gamma|^2) ds + 2\|k\|_\infty^3 \int_\gamma |\gamma| ds + \|k\|_\infty^4 \int_\gamma |\gamma|^2 ds \\ &\leq - \int_\gamma k_s(-2k\langle \gamma, \tau \rangle + k_s|\gamma|^2 + 2k\langle \gamma, \tau \rangle) ds \\ &\quad + C^2 L^{-2} + C^4 L^{-4} \int_\gamma |\gamma|^2 ds \\ &\leq - \int_\gamma k_s^2 |\gamma|^2 ds + C^2 |e|^{-2} + C^4 |e|^{-4} \int_\gamma |\gamma|^2 ds. \end{aligned}$$

Therefore again by the Grönwall inequality we obtain an estimate uniform on the finite time interval $[0, T]$. In order to upgrade this to a pointwise estimate we note that

$$(17) \quad \|\gamma\|_\infty \leq \frac{1}{L} \int_\gamma |\gamma| ds + \frac{L}{2} \leq \frac{L_0^{\frac{1}{2}}}{|e|} \|\gamma\|_2 + \frac{L_0}{2}.$$

Thus neither $\|\gamma\|_\infty$ nor $\|k_s\|_2$ may blow up, implying the result. \square

The interpolation method can be used to show bounds for all derivatives of curvature in L^2 that are globally uniform.

Lemma 3.7. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that*

$$L(\gamma_0) \int_\gamma k^2 ds \Big|_{t=0} \leq \pi.$$

Then $T = \infty$ and there exists absolute constants c_l such that

$$\|k_{s^l}\|_2^2 \leq c_l L^{-2l-1}.$$

Proof. Using Lemma 2.3 for the evolution of k_{s^l} and Lemma 2.6 to eliminate boundary terms, we find

$$\begin{aligned}
\frac{d}{dt} \int_{\gamma} k_{s^l}^2 ds &= \frac{1}{2} \int_{\gamma} 4k_{s^l} \partial_t k_{s^l} + k_{s^l}^2 (2kk_{ss} + k^4) ds \\
&= \frac{1}{2} \int_{\gamma} k_{s^l}^2 (2kk_{ss} + k^4) ds - 2 \int_{\gamma} k_{s^l} k_{s^{(l+4)}} ds \\
&\quad + \sum_{q+r+u=l} c_{qru} \int_{\gamma} k_{s^l} k_{s^{(q+2)}} k_{s^r} k_{s^u} ds \\
&\quad + \sum_{q+r+u+v+w=l} c_{qruvw} \int_{\gamma} k_{s^l} k_{s^q} k_{s^r} k_{s^u} k_{s^v} k_{s^w} ds \\
&= -2 \int_{\gamma} k_{s^{(l+2)}}^2 ds + \sum_{q+r+u=l} c_{qru} \int_{\gamma} k_{s^l} k_{s^{(q+2)}} k_{s^r} k_{s^u} ds \\
&\quad + \sum_{q+r+u+v+w=l} c_{qruvw} \int_{\gamma} k_{s^l} k_{s^q} k_{s^r} k_{s^u} k_{s^v} k_{s^w} ds.
\end{aligned}$$

Interpolating again with [18, Proposition 2.5] (we note again that [14, Appendix C] can be consulted for full details), we find

$$\frac{d}{dt} \int_{\gamma} k_{s^l}^2 ds + \int_{\gamma} k_{s^{(l+2)}}^2 ds \leq C$$

where C depends on the upper and lower bounds for length, as well as the bound for curvature. As before, this implies the differential inequality

$$\frac{d}{dt} \int_{\gamma} k_{s^l}^2 ds + \hat{c} \int_{\gamma} k_{s^{(l)}}^2 ds \leq \hat{C}$$

which implies $\|k_{s^l}\|_2^2$ is uniformly bounded on $[0, \infty)$. Rescaling yields the result. \square

With the stronger hypothesis used in Lemma 3.4, convergence to a straight line is obtained by explicit decay estimate. Here we have $T = \infty$ however we have no decay estimate. In order to obtain convergence we first find a convergent subsequence. This relies on the variational structure of the flow.

Proposition 3.8. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) given by Theorem 1.1. Assume that*

$$L(\gamma_0) \int_{\gamma} k^2 ds \Big|_{t=0} \leq \pi.$$

Then there exists a subsequence of times $\{t_j\}$ such that

$$\gamma(\cdot, t_j) \rightarrow \gamma_{\infty}$$

uniformly in C^2 with γ_{∞} a straight line.

Proof. Global existence implies

$$\int_0^{\infty} \int_{\gamma} F^2 ds dt = \|k\|_2^2|_{t=0} - \|k\|_2^2 \leq \|k\|_2^2|_{t=0}$$

so there exists a subsequence $\{t_i\} \subset [0, \infty)$ such that $\int_{\gamma} |F(\gamma(\cdot, t_i))|^2 ds \rightarrow 0$ as $i \rightarrow \infty$.

For any $\varepsilon > 0$, $\int_{\gamma} F^2 ds$ is eventually smaller than ε . Let T_0, N be such that for $i > N$ we have $t_i > T_0$ and

$$\int_{\gamma} F^2(\gamma(\cdot, t_i)) ds < \varepsilon.$$

Then

$$\frac{1}{2} \int_{\gamma} k^3 ds = \int_{\gamma} F ds \leq \sqrt{L} \|F\|_2 \leq \sqrt{L\varepsilon}.$$

In particular we have the estimate

$$K_{osc}(\gamma(\cdot, t_i)) \leq L^{\frac{1}{3}} \|k\|_3^2 \leq (4L\varepsilon)^{\frac{1}{3}}.$$

By setting $\varepsilon = \frac{(4\pi/7)^3}{4|e|}$, we see that after some time the hypothesis of Lemma 3.4 is satisfied. Then $\|k_s\|_2^2(t)$ decays and becomes eventually arbitrarily small *on the full sequence*. In particular it remains uniformly bounded on unbounded time intervals.

All of this implies there exists a limit curve γ_{∞} that the flow subconverges to which satisfies $k = 0$, that is, γ_{∞} is a straight line. \square

It remains to identify a particular line that the flow converges to, that is, we must prove that the limit is unique.

Theorem 3.9 (Theorem 1.4). *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (E) in case (b) given by Theorem 1.1. Assume that*

$$L(\gamma_0) \int_{\gamma} k^2 ds \Big|_{t=0} \leq \pi.$$

Then the flow exists globally ($T = \infty$) and converges smoothly to a straight line.

Proof. As we have proven that the derivative of curvature decays to zero in L^2 along the flow, we are squarely in the realm of classical Lyapunov stability. Standard theory implies that the flow converges to a solution of $k(\gamma_{\infty}) = 0$.

The only issue is that this does not uniquely determine the limit. Decay in the curvature implies that for large times we may write the flow as a fourth-order quasilinear scalar problem for a graph function $u : (-|e|/2, |e|/2) \times [T_0, \infty) \rightarrow \mathbb{R}$ with Neumann boundary condition. (Note that here we have used a tangential diffeomorphism to change the interval $(-1, 1)$ to $(-|e|/2, |e|/2)$.)

In this setting we apply Hale-Raugel [29] (see also Matano [43] and Zelenyak [69]). We note that a similar application was recently made in [21] for another fourth-order parabolic problem.

In our case we need to check that if the ω -limit set of a precompact orbit belongs to the space of all equilibria then the linearisation around any equilibrium point has zero as an eigenvalue with multiplicity at most one. The conclusion from [29] is that the limit of the flow is then unique.

Let $l : (-|e|/2, |e|/2) \rightarrow \mathbb{R}$ be a straight line such that for a sequence $\{t_j\} \subset [0, \infty)$ we have $\gamma(\cdot, t_j) \rightarrow l$ in C^2 . By composition with a single rotation and translation we may assume that $l(x) = (x, 0)$.

Consider $\phi = l + \varepsilon \eta$. The first boundary condition is $(\langle \phi_s, e_2 \rangle)(\pm|e|/2) = 0$, implying $\eta_x^2(\pm|e|/2) = 0$. Here we have used a superscript to denote component, so that $X^1 = \langle X, e_1 \rangle$ and $X^2 = \langle X, e_2 \rangle$.

For the other boundary condition, we first note that

$$\partial_s = \frac{1}{|l_x + \varepsilon\eta_x|} \partial_x$$

we have $\partial_x |l_x + \varepsilon\eta_x|^2 = 2 \langle e_1 + \varepsilon\eta_x, \varepsilon\eta_{xx} \rangle$ and

$$\begin{aligned} \kappa^\phi &= \phi_{ss} \\ &= \frac{1}{|l_x + \varepsilon\eta_x|} \partial_x \left(\frac{l_x + \varepsilon\eta_x}{|l_x + \varepsilon\eta_x|} \right) \\ &= \frac{\varepsilon\eta_{xx}|e_1 + \varepsilon\eta_x| - \langle e_1 + \varepsilon\eta_x, \varepsilon\eta_{xx} \rangle (e_1 + \varepsilon\eta_x)}{|e_1 + \varepsilon\eta_x|^3}. \end{aligned}$$

Then

$$\begin{aligned} k^\phi &= \langle \phi_{ss}, \text{rot}_{\pi/2}(\phi_s) \rangle \\ &= \left\langle \frac{\varepsilon\eta_{xx}|e_1 + \varepsilon\eta_x| - \langle e_1 + \varepsilon\eta_x, \varepsilon\eta_{xx} \rangle (e_1 + \varepsilon\eta_x)}{|e_1 + \varepsilon\eta_x|^3}, \frac{\text{rot}_{\pi/2}(e_1 + \varepsilon\eta_x)}{|e_1 + \varepsilon\eta_x|} \right\rangle \\ (18) \quad &= \varepsilon \frac{\langle \eta_{xx}, \text{rot}_{\pi/2}(e_1 + \varepsilon\eta_x) \rangle}{|e_1 + \varepsilon\eta_x|^3}, \end{aligned}$$

and

$$\begin{aligned} k_s^\phi &= \varepsilon \frac{1}{|e_1 + \varepsilon\eta_x|} \partial_x \frac{\langle \eta_{xx}, \text{rot}_{\pi/2}(e_1 + \varepsilon\eta_x) \rangle}{|e_1 + \varepsilon\eta_x|^3} \\ &= \varepsilon \frac{\langle \eta_{x^3}, \text{rot}_{\pi/2}(e_1 + \varepsilon\eta_x) \rangle + \varepsilon \langle \eta_{xx}, \text{rot}_{\pi/2}(\eta_{xx}) \rangle}{|e_1 + \varepsilon\eta_x|^4} - 3\varepsilon \frac{\langle \eta_{xx}, \text{rot}_{\pi/2}(e_1 + \varepsilon\eta_x) \rangle \langle e_1 + \varepsilon\eta_x, \varepsilon\eta_{xx} \rangle}{|e_1 + \varepsilon\eta_x|^5} \\ &= \varepsilon \frac{\langle \eta_{x^3}, \text{rot}_{\pi/2}(e_1 + \varepsilon\eta_x) \rangle}{|e_1 + \varepsilon\eta_x|^4} - 3\varepsilon \frac{\langle \eta_{xx}, \text{rot}_{\pi/2}(e_1 + \varepsilon\eta_x) \rangle \langle e_1 + \varepsilon\eta_x, \varepsilon\eta_{xx} \rangle}{|e_1 + \varepsilon\eta_x|^5}. \end{aligned}$$

Therefore the boundary condition $k_s^\phi(\pm|e|/2) = 0$ reduces to (note that $\eta_x^2(\pm|e|/2) = 0$)

$$\left(\eta_{x^3}^2(1 + \varepsilon\eta_x^1) - 3\varepsilon\eta_{xx}^1\eta_{xx}^2|1 + \varepsilon\eta_x^1| \right) (\pm|e|/2) = 0.$$

Now we compute the linearisation of F , given by

$$F[\gamma] = \langle \gamma_{ss}, \nu \rangle_{ss} + \frac{1}{2} \langle \gamma_{ss}, \nu \rangle^3.$$

First we calculate the commutator

$$\partial_\varepsilon \partial_s = \partial_s \partial_\varepsilon - \frac{\langle e_1 + \varepsilon\eta_x, \eta_x \rangle}{|e_1 + \varepsilon\eta_x|^2} \partial_s.$$

Note that as $\varepsilon \searrow 0$ it is not the case that $[\partial_\varepsilon, \partial_s] \rightarrow 0$. Now

$$\begin{aligned}
\partial_\varepsilon F(l + \varepsilon \eta) &= \partial_\varepsilon \left(\partial_s^2 k^\phi + \frac{1}{2} (k^\phi)^3 \right) \\
&= \partial_s \partial_\varepsilon (\partial_s k^\phi) - \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s^2 k^\phi + \frac{3}{2} (k^\phi)^2 \partial_\varepsilon k^\phi \\
&= \partial_s \left(\partial_s \partial_\varepsilon k^\phi - \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s k^\phi \right) \\
&\quad - \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s^2 k^\phi + \frac{3}{2} (k^\phi)^2 \partial_\varepsilon k^\phi \\
&= \partial_s^2 \partial_\varepsilon k^\phi - \partial_s \left(\frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \right) \partial_s k^\phi \\
&\quad - 2 \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s^2 k^\phi + \frac{3}{2} (k^\phi)^2 \partial_\varepsilon k^\phi.
\end{aligned}$$

Although we have used along the way the fact that l is a straight line with $l_s = e_1$ and $l_{ss} = 0$, now is where it really becomes incredibly special. We have

$$k^\phi|_{\varepsilon=0} = 0, \quad k_s^\phi|_{\varepsilon=0} = 0, \quad \text{and} \quad k_{ss}^\phi|_{\varepsilon=0} = 0.$$

From (18) we calculate

$$\partial_\varepsilon k^\phi = \frac{\langle \eta_{xx}, \text{rot}_{\pi/2}(e_1 + \varepsilon \eta_x) \rangle}{|e_1 + \varepsilon \eta_x|^3} + \varepsilon \partial_\varepsilon \frac{\langle \eta_{xx}, \text{rot}_{\pi/2}(e_1 + \varepsilon \eta_x) \rangle}{|e_1 + \varepsilon \eta_x|^3},$$

therefore

$$\partial_\varepsilon k^\phi|_{\varepsilon=0} = \eta_{xx}^2.$$

Plugging everything in, we find

$$\partial_\varepsilon F(l + \varepsilon \eta)|_{\varepsilon=0} = \eta_{xx}^4$$

Therefore, converting finally to the graph setting where curves are represented by graphs $(x, u(x))$ of functions $u : (-|e|/2, |e|/2) \rightarrow \mathbb{R}$, we have

$$\mathcal{L}(u) = u_{xx}^4$$

for the linearisation of F about a straight line, in the graph setting, with boundary conditions $\eta_x(\pm 1) = 0$ and $\eta_{xx}(\pm 1) = 0$.

The eigenvalues are all negative apart from a one-dimensional eigenspace corresponding to constants, i.e. vertical translation, as expected. Therefore Hale-Raugel's theorem applies and we are done. \square

Remark 8 (Łojasiewicz-Simon). There are by now many ways to obtain convergence from a position such as this one. The approach of Simon [54], using in particular the Łojasiewicz-Simon inequality, can be used to obtain full convergence of the flow. Simon's original application was to study the asymptotics of second-order quasilinear parabolic systems on Riemannian manifolds with application to minimal submanifold theory. However the technique has incredible generality (Simon's work is itself an infinite-dimensional generalisation of Łojasiewicz's work on analytic functions) and is by now used in all areas of parabolic PDE, including our setting here of fourth-order problems with boundary.

For the elastic flow, all candidate limits have zero energy. They are not separated in standard function spaces, being continuous in the C^0 -norm for example. Movement in a direction perpendicular to e corresponds to a degeneracy in the differential operator.

There are two strategies to prove the Łojasiewicz-Simon inequality that we wish to describe. First, as we have subconvergence to a straight line γ_∞ , we may (as in the proof given above) use this line to describe the flow at later times as the graph of a function $u : [-1, 1] \rightarrow \mathbb{R}$, that is,

$$(x, u(x)) = \gamma(x)$$

where γ may have undergone reparametrisation by a tangential diffeomorphism. The flow reduces to a fourth-order quasilinear scalar PDE with Neumann boundary conditions. The Łojasiewicz-Simon inequality has been established for similar PDE in this setting by for example Rybka-Hoffmann [53]. A similar argument works here.

The other strategy which we wish to describe is due to Chill [12], who provides three generic criteria², which we translate here as:

- That the energy is analytic;
- That the gradient of the energy is analytic;
- That the derivative of the gradient of the energy evaluated at zero is Fredholm.

Of course, each of the above need to be understood in appropriate function spaces. Recently, Dall'Acqua-Pozzi-Spener [15] have, for a constrained variation of the elastic flow, proved each of the above with clamped boundary conditions. In light of Lemma 2.6, their proof applies also here, and so this approach again yields the Łojasiewicz-Simon inequality. We did not use the Łojasiewicz-Simon inequality to obtain convergence for (E), as the above elementary linearisation argument works nicely.

Remark 9 (Novaga-Okabe). A new approach to obtaining full convergence from subconvergence is the innovative technique of Novaga-Okabe [48]. They prove that with certain boundary conditions the set of equilibria are separated in the plane in the sense of the Hausdorff metric, and then show that in a very general setting this implies full convergence.

The separation condition is at the level of *energy*, and so for the method to apply these continuous actions also need to be invariants on the level of energy.

For the flows (CD) and (E) translation acts continuously on equilibria and is an invariant of the respective energies. Therefore this technique does not appear to work in these cases. However, for other kinds of boundary conditions that rule out such continuous actions, we expect that their technique may prove efficient.

Let us return to considering the curve diffusion flow. We hope to prove that the flow converges to a straight line. This equilibrium has zero curvature, and so by Lemma 2.5, any solution that is asymptotic to a straight line must have $\bar{k}(\gamma_t) = 0$ for all t . Now we indirectly assume that $\omega = 0$ by taking on an assumption such as (4). Then $K_{osc} = L\|k\|_2^2$, and a-priori estimates become much simpler. This is illustrated below.

Lemma 3.10. *Let $\gamma : (-1, 1) \times [0, T] \rightarrow \mathbb{R}^2$ be a solution to (CD) given by Theorem 1.1 with $\omega = 0$. Then, if*

$$L(\gamma_0)\|k\|_2^2(\gamma_0) \leq \frac{2\pi}{3}$$

²Chill's argument applies more generally; for full details, we invite the interested reader to consult [12].

we have the uniform a-priori estimate

$$L(\gamma_t)\|k\|_2^2(\gamma_t) \leq L(\gamma_0)\|k\|_2^2(\gamma_0)$$

for all $t \in [0, T]$.

Proof. Using Lemma 2.6 we have

$$\begin{aligned} \frac{d}{dt}K_{osc} + K_{osc}\frac{\|k_s\|_2^2}{L} + 2L\|k_{ss}\|_2^2 \\ = 3L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds + 6\bar{k}L \int_{\gamma} (k - \bar{k}) k_s^2 ds + 2\bar{k}^2 L \|k_s\|_2^2. \end{aligned}$$

Since $\bar{k} = 0$, this simplifies to

$$\frac{d}{dt}(L\|k\|_2^2) + \|k\|_2^2\|k_s\|_2^2 + 2L\|k_{ss}\|_2^2 = 3L \int_{\gamma} k^2 k_s^2 ds.$$

This term is estimated by

$$3L \int_{\gamma} k^2 k_s^2 ds \leq \frac{3L}{\pi} (L\|k\|_2^2) \|k_{ss}\|_2^2.$$

Therefore

$$\frac{d}{dt}(L\|k\|_2^2) + \|k\|_2^2\|k_s\|_2^2 + L\left(2 - \frac{3}{\pi}L\|k\|_2^2\right)\|k_{ss}\|_2^2 \leq 0,$$

which upon integration yields

$$L\|k\|_2^2(\gamma_t) + \int_0^t \|k\|_2^2\|k_s\|_2^2 d\tau \leq L\|k\|_2^2(\gamma_0),$$

as required. \square

Theorem 1.1 tells us that in order to conclude global existence, it is enough to find an a-priori estimate for the position vector in L^∞ and the first derivative of curvature in L^2 . Our strategy for this is to show first an a-priori estimate for $\|k_{ss}\|_2$, the velocity in L^2 .

Lemma 3.11. *Let $\gamma : (-1, 1) \times [0, T] \rightarrow \mathbb{R}^2$ be a solution to (CD) given by Theorem 1.1. Assume that*

$$L(\gamma_0)\|k\|_2^2(\gamma_0) < \frac{\pi}{10}.$$

Then for a universal constant $C_0 \in (0, \infty)$ we have the a-priori estimate

$$(19) \quad \left| \int_{\gamma} k_{ss}^2 ds \right|_t \leq C_0.$$

Furthermore, we have for some uniform constant $\delta_0 > 0$ the exponential decay

$$(20) \quad \left| \int_{\gamma} k_{ss}^2 ds \right|_t \leq \left| \int_{\gamma} k_{ss}^2 ds \right|_{t=0} e^{-\delta_0 t}.$$

Proof. Let us first note that

$$(21) \quad -2 \int_{\gamma} k_{ss} k_{ss} ds = -2\|k_{ss}\|_2^2.$$

To see this, we compute

$$\begin{aligned} -2 \int_{\gamma} k_{s^6} k_{ss} ds &= [-2k_{s^5} k_{ss}]_{\{0,L\}} + 2 \int_{\gamma} k_{s^5} k_{s^3} ds \\ &= -2[k_{s^5} k_{ss}]_{\{0,L\}} + 2[k_{s^4} k_{s^3}]_{\{0,L\}} - 2 \int_{\gamma} k_{s^4}^2 ds. \end{aligned}$$

Lemma 2.6 implies that the first two terms vanish, and therefore (21) follows.

Let us now compute the evolution of $\|k_{ss}\|_2^2$. By using the commutator $[\partial_s, \partial_t]$ as in Lemma 2.2, we find

$$\begin{aligned} k_{sst} &= k_{sts} - k k_{ss}^2 \\ &= (-k_{s^5} - k^2 k_{s^3} - 3k_{ss} k_s k)_s - k k_{ss}^2 \\ &= -k_{s^6} - k^2 k_{s^4} - 3k_{s^3} k_s k - k k_{ss}^2 - 2k k_s k_{s^3} - 3k_{ss} (k_{ss} k + k_s^2) \\ &= -k_{s^6} - k^2 k_{s^4} - 5k_{s^3} k_s k - 4k k_{ss}^2 - 3k_{ss} k_s^2. \end{aligned}$$

Using this we compute

$$\begin{aligned} \frac{d}{dt} \int_{\gamma} k_{ss}^2 ds &= \int_{\gamma} [-2k_{s^6} k_{ss} - 2k^2 k_{s^4} k_{ss} - 10k_{s^3} k_s k_{ss} k - 7k k_{ss}^3 - 6k_{ss}^2 k_s^2] ds \\ &= -2\|k_{s^4}\|_2^2 - 6\|k_s k_{ss}\|_2^2 + \int_{\gamma} [-2k_{s^4} k_{ss} k^2 - 10k_{s^3} k_s k_{ss} k - 7k_{ss}^3 k] ds \\ &= -2\|k_{s^4}\|_2^2 - 6\|k_s k_{ss}\|_2^2 + 2\|k_{s^3} k\|_2^2 + \int_{\gamma} [-3(k_{ss}^2)_s k_s k - 7k_{ss}^3 k] ds \\ &= -2\|k_{s^4}\|_2^2 - 6\|k_s k_{ss}\|_2^2 + 2\|k_{s^3} k\|_2^2 + \int_{\gamma} 3k_{ss}^3 k + 3k_{ss}^2 k_s^2 - 7k_{ss}^3 k ds \\ &= -2\|k_{s^4}\|_2^2 - 3\|k_s k_{ss}\|_2^2 + 2\|k_{s^3} k\|_2^2 - \int_{\gamma} 4k_{ss}^3 k ds. \end{aligned}$$

In the third equality above we used that k_{s^3} vanishes on the boundary (Lemma 2.6), and in the fourth equality we used the no-flux condition.

Now let us estimate

$$\begin{aligned} \int_{\gamma} k_{s^3}^2 k^2 ds &= \int_{\gamma} k_{s^3}^2 (k - \bar{k})^2 ds - \bar{k}^2 \int_{\gamma} k_{s^3}^2 ds + 2\bar{k} \int_{\gamma} k_{s^3}^2 k ds \\ &\leq \frac{1}{2} \int_{\gamma} k_{s^3}^2 k^2 ds + \bar{k}^2 \int_{\gamma} k_{s^3}^2 ds + \int_{\gamma} k_{s^3}^2 (k - \bar{k})^2 ds \end{aligned}$$

so

$$(22) \quad \int_{\gamma} k_{s^3}^2 k^2 ds \leq 2\bar{k}^2 \int_{\gamma} k_{s^3}^2 ds + 2 \int_{\gamma} k_{s^3}^2 (k - \bar{k})^2 ds.$$

For the first integral, combine

$$\int_{\gamma} k_{s^3}^2 ds = - \int_{\gamma} k_{s^4} k_{ss} ds \leq \delta_1 \int_{\gamma} k_{s^4}^2 ds + \frac{1}{4\delta_1} \int_{\gamma} k_{ss}^2 ds$$

with

$$\int_{\gamma} k_{ss}^2 ds = - \int_{\gamma} k_{s^3} (k - \bar{k})_s ds = \int_{\gamma} k_{s^4} (k - \bar{k}) ds \leq \delta_2 \int_{\gamma} k_{s^4}^2 ds + \frac{1}{4\delta_2} \frac{K_{osc}}{L}$$

to get

$$2\bar{k}^2 \int_{\gamma} k_{s^3}^2 ds \leq 2\bar{k}^2 \left(\delta_1 + \frac{\delta_2}{4\delta_1} \right) \int_{\gamma} k_{s^4}^2 ds + 2\bar{k}^2 \frac{1}{16\delta_1\delta_2} \frac{K_{osc}}{L}.$$

Choosing $\delta_1 = \frac{\varepsilon}{2\bar{k}^2}$ and $\delta_2 = \frac{\varepsilon^2}{\bar{k}^4}$ (note that $\bar{k} > 0$) we find

$$(23) \quad 2\bar{k}^2 \int_{\gamma} k_{s^3}^2 ds \leq 2\varepsilon \int_{\gamma} k_{s^4}^2 ds + \frac{\bar{k}^8}{4\varepsilon^3} \frac{K_{osc}}{L}.$$

The estimate (23) deals with the first term on the right of (22). The second is simple:

$$(24) \quad 2 \int_{\gamma} k_{s^3}^2 (k - \bar{k})^2 ds \leq 2 \|k_{s^3}\|_{\infty}^2 \frac{K_{osc}}{L} \leq 2 \frac{K_{osc}}{\pi} \|k_{s^4}\|_2^2.$$

Combining (23), (24) and (22) we find

$$(25) \quad \int_{\gamma} k_{s^3}^2 k^2 ds \leq 2 \left(\varepsilon + \frac{K_{osc}}{\pi} \right) \int_{\gamma} k_{s^4}^2 ds + \frac{\bar{k}^8}{4\varepsilon^3} \frac{K_{osc}}{L}.$$

Now let us estimate

$$\begin{aligned} -4 \int_{\gamma} k_{ss}^3 k ds &= 4 \int_{\gamma} k_{ss}^2 k_s^2 ds + 8 \int_{\gamma} k_{s^3} k_{ss} k_s k ds \\ &\leq 8 \int_{\gamma} k_{ss}^2 k_s^2 ds + 4 \int_{\gamma} k_{s^3}^2 k^2 ds. \end{aligned}$$

The second term is estimated by (25). For the first,

$$\begin{aligned} 6 \int_{\gamma} k_{ss}^2 k_s^2 ds &\leq \frac{L}{\pi} \left(\int_{\gamma} k_{ss}^2 ds \right)^2 \\ &= \frac{L}{\pi} \left(\int_{\gamma} k_{s^4} (k - \bar{k}) ds \right)^2 \\ &\leq \frac{K_{osc}}{\pi} \int_{\gamma} k_{s^4}^2 ds. \end{aligned}$$

Combining these estimates, we find

$$\begin{aligned} \frac{d}{dt} \int_{\gamma} k_{ss}^2 ds + 3 \|k_s k_{ss}\|_2^2 + 2 \|k_{s^4}\|_2^2 &= 2 \|k_{s^3} k\|_2^2 - 4 \int_{\gamma} k_{ss}^3 k ds \\ &\leq (2+4) \left[2 \left(\varepsilon + \frac{K_{osc}}{\pi} \right) \int_{\gamma} k_{s^4}^2 ds + \frac{\bar{k}^8}{4\varepsilon^3} \frac{K_{osc}}{L} \right] + 8 \left[\frac{K_{osc}}{\pi} \int_{\gamma} k_{s^4}^2 ds \right] \end{aligned}$$

or

$$(26) \quad \frac{d}{dt} \int_{\gamma} k_{ss}^2 ds + 3 \|k_s k_{ss}\|_2^2 + \left(2 - 12\varepsilon - 20 \frac{K_{osc}}{\pi} \right) \|k_{s^4}\|_2^2 \leq 5 \frac{\bar{k}^8}{4\varepsilon^3} \frac{K_{osc}}{L}.$$

Now we use that $\omega = 0$ to simplify estimate (26) reads

$$(27) \quad \frac{d}{dt} \int_{\gamma} k_{ss}^2 ds + 2 \|k_s k_{ss}\|_2^2 + \left(2 - 12\varepsilon - 20 \frac{K_{osc}}{\pi} \right) \|k_{s^4}\|_2^2 \leq 0.$$

By hypothesis Lemma 3.10 applies and we have

$$K_{osc}(\gamma_t) < \frac{\pi}{10}$$

for all t . This implies that for some $\delta > 0$

$$2 - 12\varepsilon - 20\frac{K_{osc}}{\pi} > \frac{20\delta}{\pi} - 12\varepsilon$$

and so choosing $\varepsilon = \frac{4\delta}{3\pi}$ we find

$$(28) \quad \frac{d}{dt} \int_{\gamma} k_{ss}^2 ds + 2\|k_s k_{ss}\|_2^2 + \frac{4\delta}{\pi} \|k_{s^4}\|_2^2 \leq 0.$$

Lemmata 2.7 and 2.8 imply

$$\|k_{s^2}\|_2^2 \leq \frac{L^2}{\pi^2} \|k_{s^3}\|_2^2 \leq \frac{L^4}{\pi^4} \|k_{s^4}\|_2^2$$

so that

$$-\|k_{s^4}\|_2^2 \leq -\frac{\pi^4}{L^4(\gamma_0)} I^2(\gamma_0) \int_{\gamma} k_{ss}^2 ds \leq C \int_{\gamma} k_{ss}^2 ds.$$

Combining this with (28) yields

$$\frac{d}{dt} \int_{\gamma} k_{ss}^2 ds \leq -\tilde{C}\delta \int_{\gamma} k_{ss}^2 ds,$$

for $\tilde{C} = C\frac{4\delta}{\pi}$. Therefore we conclude the exponential decay estimate (20). \square

Lemma 3.12. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD) given by Theorem 1.1. Under the assumptions of Lemma 3.11, for a universal constant $C \in (0, \infty)$ we have*

$$(29) \quad \|\gamma\|_{\infty}^2 \leq C(1+t)e^t.$$

Proof. We calculate

$$(30) \quad \frac{d}{dt} \int_{\gamma} |\gamma|^2 ds = -2 \int_{\gamma} \langle \gamma, k_{ss} \nu \rangle ds + \int_{\gamma} |\gamma|^2 k k_{ss} ds.$$

First, Lemma 3.11 allows us to uniformly estimate

$$(31) \quad -2 \int_{\gamma} \langle \gamma, k_{ss} \nu \rangle ds \leq \int_{\gamma} k_{ss}^2 ds + \int_{\gamma} |\gamma|^2 ds \leq C + \int_{\gamma} |\gamma|^2 ds.$$

Now let us deal with the second term. Integrating by parts and estimating, we find

$$\begin{aligned} \int_{\gamma} |\gamma|^2 k k_{ss} ds &= - \int_{\gamma} k_s^2 |\gamma|^2 ds - 2 \int_{\gamma} k k_s \langle \gamma, \tau \rangle ds \\ &\leq \int_{\gamma} k^2 ds \\ &= \int_{\gamma} (k - \bar{k})^2 ds - L\bar{k}^2 + 2L\bar{k}^2 \\ &= L^{-1}K_{osc} + L\bar{k}^2 \leq C. \end{aligned}$$

For the last estimate, we used Lemma 2.5, noting that we have $\bar{k}(\gamma_t) = 0$ (by Lemma 2.5) and Lemma 3.10.

Combining this with (30), (31), we find

$$\frac{d}{dt} \int_{\gamma} |\gamma|^2 ds \leq C + \int_{\gamma} |\gamma|^2 ds$$

which implies by Gronwall's inequality

$$(32) \quad \int_{\gamma} |\gamma|^2 ds \leq C(1+t)e^t.$$

Set $\gamma_i = \langle \gamma, e_i \rangle$. To conclude the L^∞ estimate we calculate

$$(33) \quad \|\gamma_i - \bar{\gamma}_i\|_\infty^2 \leq \frac{L}{\pi} \int_{\gamma} |\langle \gamma, e_i \rangle|^2 ds \leq \frac{L^2}{\pi}.$$

Since $\gamma_i = \gamma_i - \bar{\gamma}_i + \bar{\gamma}_i$ we conclude from the above that

$$(34) \quad \|\gamma_i\|_\infty^2 \leq \frac{L^2}{\pi} + \bar{\gamma}_i^2.$$

The last term is estimated by

$$(35) \quad \bar{\gamma}_i^2 = \frac{1}{L^2} \left(\int_{\gamma} |\langle \gamma, e_i \rangle| ds \right)^2 \leq \frac{1}{L} \int_{\gamma} |\gamma|^2 ds.$$

Since $|\gamma|^2 = \sum_{i=1}^2 \gamma_i^2$, combining (32)–(35) yields (29). \square

Now we contradict finite maximal time.

Corollary 3.13. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD) given by Theorem 1.1 satisfying the assumptions of Lemma 3.11. Then $T = \infty$.*

Proof. Suppose $T < \infty$. Then by Theorem 1.1 there does not exist a $D \in (0, \infty)$ such that

$$(36) \quad \|\gamma\|_\infty(t) + \|k_s\|_2(t) \leq D.$$

for all $t \in [0, T)$. Lemma 3.11 implies

$$\|k_s\|_2^2 \leq \frac{L^2}{\pi^2} \|k_{ss}\|_2^2 \leq C_1.$$

This deals with the second term. Lemma 3.12 implies

$$\|\gamma\|_\infty < C_2$$

where $C_2 = C(1+T)e^T$. These estimates imply that (36) holds for $D = C_1 + C_2$, yielding the desired contradiction. Therefore $T = \infty$. \square

Corollary 3.14. *Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD) given by Theorem 1.1 satisfying the assumptions of Lemma 3.11. Then $T = \infty$ and $\gamma(\cdot, t)$ approaches a straight line parallel to e .*

Proof. Corollary 3.13 implies $T = \infty$, and Lemma 3.1 gives $K_{osc} \in L^1([0, \infty))$, so if K'_{osc} is bounded, we shall be able to directly conclude $K_{osc} \rightarrow 0$ and obtain the desired

statement. Recall Lemma 2.10 and estimate

$$\begin{aligned} \left| \frac{d}{dt} K_{osc} \right| &\leq 3L \int_{\gamma} (k - \bar{k})^2 k_s^2 ds + 6\bar{k}L \int_{\gamma} |k - \bar{k}| k_s^2 ds + 2\bar{k}^2 L \|k_s\|_2^2 \\ &\leq C \|k_s\|_2^4 + C \|k_s\|_2^3 + C \|k_s\|_2^2 \\ &\leq C, \end{aligned}$$

where we used that length is non-increasing along the flow, the Poincaré-Wirtinger inequalities, and the a-priori estimate Lemma 3.11. Note that the constant C above is an absolute constant, depending only on γ_0 . This finishes the proof. \square

We now wish to conclude from Corollary 3.14 exponentially fast convergence in C^∞ . For this we use the same linearisation idea as before.

Theorem 3.15 (Theorem 1.2). *Suppose $|e| > 0$. Let $\gamma : (-1, 1) \times [0, T) \rightarrow \mathbb{R}^2$ be a solution to (CD). Suppose γ_0 satisfies*

$$L(\gamma_0) \|k\|_2^2(\gamma_0) < \frac{\pi}{10}.$$

Then $\omega = 0$, the flow exists globally $T = \infty$, and $\gamma(\cdot, t)$ converges exponentially fast to a translate of e in the C^∞ topology.

Proof. Given our work above, the proof of Theorem 1.4 applies almost verbatim in this situation. The only difference is that for (CD)

$$F[\gamma] = \langle \gamma_{ss}, \nu \rangle_{ss}.$$

Then

$$\begin{aligned} \partial_\varepsilon F(l + \varepsilon \eta) &= \partial_\varepsilon \left(\partial_s^2 k^\phi \right) \\ &= \partial_s \partial_\varepsilon (\partial_s k^\phi) - \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s^2 k^\phi \\ &= \partial_s \left(\partial_s \partial_\varepsilon k^\phi - \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s k^\phi \right) \\ &\quad - \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s^2 k^\phi \\ &= \partial_s^2 \partial_\varepsilon k^\phi - \partial_s \left(\frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \right) \partial_s k^\phi \\ &\quad - 2 \frac{\langle e_1 + \varepsilon \eta_x, \eta_x \rangle}{|e_1 + \varepsilon \eta_x|^2} \partial_s^2 k^\phi. \end{aligned}$$

As l is straight, we have

$$k^\phi|_{\varepsilon=0} = 0, \quad k_s^\phi|_{\varepsilon=0} = 0, \quad \text{and} \quad k_{ss}^\phi|_{\varepsilon=0} = 0.$$

Therefore

$$\partial_\varepsilon F(l + \varepsilon \eta)|_{\varepsilon=0} = \eta_{x^4}^2$$

This is the same linearisation as with the elastic flow, and the argument continues identically from here to that of Theorem 1.4. \square

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